New tests and applications of the worldline path integral in the first order formalism

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Abstract

We present different non-perturbative calculations within the context of Migdal's representation for the propagator and effective action of quantum particles. We first calculate the exact propagators and effective actions for Dirac, scalar and Proca fields in the presence of constant electromagnetic fields, for an even-dimensional spacetime. Then we derive the propagator for a charged scalar field in a spacelike vortex (i.e., instanton) background, in a long-distance expansion, and the exact propagator for a massless Dirac field in $1+1$ dimensions in an arbitrary background. Finally, we present an interpretation of the chiral anomaly in the present context, finding a condition that the paths must fulfil in order to have a non-vanishing anomaly.

1 Introduction

Worldline formulations have been applied since a long time ago [1] to the derivation of many interesting Quantum Field Theory results. More recent applications have emerged as a by-product of the new insights gained by the re-derivation of worldline representations by taking the infinite tension limit in (perturbative) string theory amplitudes [2], and by the introduction of new ways to handle the spin degrees of freedom [3, 4, 5]. Besides, elegant proposals

to treat more general situations, involving internal degrees of freedom and general couplings to higher-spin fields have been advanced [6, 7].

In these methods, different sets of variables and alternative constructions have been used in order to 'exponentiate' the relevant observables and then perform the path integral. In spite of the *formal* equivalence between the different methods, there are few concrete calculations that may serve as tests to gain a deeper understanding of the method and about the physics involved. Important steps in that direction have already been taken; indeed, some non-perturbative calculations corresponding to external constant electromagnetic fields have been obtained within the worldline representation [8]. Another setting were the worldline approach can be independently tested is in numerical calculations [9].

In this article, we present new tests corresponding to concrete examples, obtained within the worldline path-integral representation for Dirac fields in the first order formalism introduced by Migdal in [10] and further extended in [4, 5]. The first order formalism preserves the geometrical picture and is quite intuitive (for example, it does not involve Grassmann variables). These two features are, we believe, among the main advantages of the worldline method. It is also more adequate for some numerical computations which are specially suited for non perturbative calculations.
We shall follow our previous work [11], where some features of the method have been discussed in detail, including a proof of the equivalence with standard Quantum Field Theory at the perturbative level.

The structure of this paper is as follows: in section 2 we briefly review the main properties of the representation introduced in [10], with emphasis in the objects we shall be concerned with in the examples. To elucidate the quite general nature of this approach, we also introduce the worldline representation for the propagator and the effective action corresponding to a Proca field.

In 3 we deal with a constant $F_{\mu\nu}$ field in $1+1$ dimensions, for the Dirac, scalar and Proca cases. In section 4, we generalize the previous cases to $d > 2$ ($d =$ even) dimensions.

Section 5 contains a derivation of the scalar propagator in a vortex-like background in $1+1$ dimensions, and section 6 the calculation of the exact propagator for a massless Dirac field in $1+1$ dimensions.

In section 7 we present a study of the kind of 'roughness' one should expect the trajectories to have, in order to contribute to the chiral anomaly. We show that there has to be a singularity in the correlation between different time derivatives of the paths, whose precise form depends on the number of spacetime dimension.

Finally, section 8 presents our conclusions.

2 The method

Our aim here is to calculate the propagator and effective action corresponding to external electromagnetic fields. We shall be concerned with Dirac, scalar and Proca models, coupled to Abelian gauge fields.

Let us consider first the case of a massive Dirac field in d Euclidean dimensions, whose action, S_f, has the following form:

$$S_f(\bar{\psi}, \psi, A) \;=\; \int d^d x \, \bar{\psi}(\not{D} + m)\psi \;, \tag{1}$$

where

$$\not{D} \equiv \gamma_\mu D_\mu \;\;,\quad D_\mu = \partial_\mu + ieA_\mu \;\;,\quad \gamma_\mu^\dagger = \gamma_\mu \;\;,\quad \mu = 1, \ldots, d \;. \tag{2}$$

The γ_μ matrices satisfy the Clifford Algebra:

$$\{\gamma_\mu, \gamma_\nu\} = 2\,\delta_{\mu\nu} \quad \forall \mu = 1, \ldots, d \;, \tag{3}$$

A_μ denotes an Abelian gauge field and e is a coupling constant with the dimensions of $[\text{mass}]^{\frac{4-d}{2}}$.

In the worldline formulation of [10], the fermion propagator, denoted here by $G_f(x,y)$, is represented by the path integral:

$$\begin{aligned} G_f(x,y) \;\; &= \;\; \int_0^\infty dT \, e^{-mT} \int_{x(0)=y}^{x(T)=x} \mathcal{D}p \mathcal{D}x \; e^{i\int_0^T d\tau p(\tau)\cdot\dot{x}(\tau)} \\ &\times \;\; \mathcal{P}\big[e^{-i\int_0^T d\tau \not{p}(\tau)}\big] \;\; e^{-ie\int_0^T d\tau \dot{x}(\tau)\cdot A[x(\tau)]} \;, \end{aligned} \tag{4}$$

where we have explicitly indicated the boundary conditions for the $x_\mu(\tau)$ paths. The $p_\mu(\tau)$ paths are, on the other hand, unconstrained.

Another object we will be interested in is $\Gamma_f(A)$, the (normalized) contribution of the fermionic determinant to the effective action:

$$\Gamma_f(A) \;\equiv\; -\ln\Big[\frac{\det\big(\not{D} + m\big)}{\det\big(\not{\partial} + m\big)}\Big] \;\;=\;\; -\mathrm{Tr}\ln\big(\not{D} + m\big) \;+\; \mathrm{Tr}\ln\big(\not{\partial} + m\big) \;, \tag{5}$$

which (by definition) verifies $\Gamma_f(0) = 0$.

For $\Gamma_f(A)$ we have the worldline representation:

$$\begin{aligned} \Gamma_f(A) \;=\; &\int_0^\infty \frac{dT}{T} \, e^{-mT} \int_{x(0)=x(T)} \mathcal{D}p \, \mathcal{D}x \; e^{i\int_0^T d\tau p_\mu(\tau)\dot{x}_\mu(\tau)} \\ &\times \mathrm{tr}\big[\mathcal{P} e^{-i\int_0^T d\tau \not{p}(\tau)}\big] \; e^{-ie\int_0^T d\tau \dot{x}_\mu(\tau) A_\mu[x(\tau)]} \;, \end{aligned} \tag{6}$$

where the functional integration measure may be formally represented as:

$$\mathcal{D}p\mathcal{D}x \;\equiv\; \prod_{0<\tau\leq T} \frac{d^dx(\tau)d^dp(\tau)}{(2\pi)^d} \;, \tag{7}$$

and it is (also formally) dimensionless, since there are as many dp's as there are dx's in the integration measure and, in our conventions, $\hbar = 1$. Of course, this formal definition can be made more rigorous by introducing a discrete approximation to it, and taking the corresponding limit. This procedure will, indeed, be used later on to deal with some examples.

We will also consider complex scalar fields, their propagators (to be denoted by G_b) and their contribution to the effective action (Γ_b). In the context of the worldline formulation we have explained before, those objects have similar expressions to their Dirac counterparts. Indeed, if the field theory action S_b for φ, $\bar{\varphi}$ is:

$$S_b \;=\; \int d^dx \left[\bar{D}_\mu\varphi D_\mu\varphi \,+\, m^2\bar{\varphi}\varphi\right] \,, \tag{8}$$

then, an entirely analogous definition to the one used for the Dirac field leads to

$$\begin{aligned} G_b(x,y) \;&=\; \int_0^\infty dT\, e^{-m^2T} \int_{x(0)=y}^{x(T)=x} \mathcal{D}p\mathcal{D}x\; e^{i\int_0^T d\tau p(\tau)\cdot\dot{x}(\tau)} \\ &\times\;\; e^{-\int_0^T d\tau\, p^2(\tau)} \;\; e^{-ie\int_0^T d\tau \dot{x}(\tau)\cdot A[x(\tau)]} \,, \end{aligned} \tag{9}$$

and

$$\begin{aligned} \Gamma_b(A) \;=\; -\int_0^\infty \frac{dT}{T}\, e^{-m^2T} \int_{x(0)=x(T)} \mathcal{D}x\mathcal{D}p\; e^{i\int_0^T d\tau p_\mu(\tau)\dot{x}_\mu(\tau)} \\ \times\, e^{-\int_0^T d\tau\, p^2(\tau)}\; e^{-ie\int_0^T d\tau \dot{x}_\mu(\tau)A_\mu[x(\tau)]} \,. \end{aligned} \tag{10}$$

It is interesting to compare the previous expressions with their Dirac field counterparts: note that the difference amounts to replacing the object

$$\Phi_f(T) \;\equiv\; \mathcal{P}\Big[e^{-i\int_0^T d\tau\, \not{p}(\tau)}\Big], \tag{11}$$

by

$$\Phi_b(T) \;\equiv\; e^{-\int_0^T d\tau\, p^2(\tau)}, \tag{12}$$

in the corresponding fermionic formula. Besides, there is a (-1) factor in Γ_b because of the different statistics; m is replaced by m^2, and the trace of Φ_b is of course absent.

This general structure will reproduce itself with more or less straightforward changes for the next example that we shall consider: the Proca field, for which we use a_μ to denote the field variable, to avoid confusion with the gauge field A_μ. The Euclidean action, S_P, is defined by:

$$S_P \;=\; \int d^dx \left(\frac{1}{4} f_{\mu\nu} f_{\mu\nu} + \frac{1}{2} m^2 a_\mu a_\mu\right) . \tag{13}$$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. This action corresponds of course to the case of a *real* field, for which it makes sense to define the (free) propagator, but to allow for a coupling to an external gauge field A_μ, we also consider the complex field version:

$$S_P(a^*, a; A) \;=\; \int d^dx \,\big(\frac{1}{2}|D_\mu a_\nu - D_\nu a_\mu|^2 + m^2 |a|^2\big) \tag{14}$$

with the covariant derivative $D_\mu = \partial_\mu + ieA_\mu$.

Based on the form of the Euclidean actions, it is rather straightforward to derive the propagator in the presence of an external field A_μ. Indeed, we have

$$\begin{aligned} G_P(x,y) \;&=\; \int_0^\infty dT \, e^{-m^2 T} \int_{x(0)=y}^{x(T)=x} \mathcal{D}p\mathcal{D}x \; e^{i\int_0^T d\tau p(\tau)\cdot\dot{x}(\tau)} \\ &\times \;\; \Phi_P(T) \;\; e^{-ie\int_0^T d\tau \dot{x}(\tau)\cdot A[x(\tau)]} \; , \end{aligned} \tag{15}$$

with

$$\Phi_P(T) \;=\; \mathcal{P} \exp\Big[- \int_0^T d\tau p_\alpha(\tau) p_\beta(\tau) \Gamma^P_{\alpha\beta}\Big] \tag{16}$$

and the $\Gamma^P_{\alpha\beta}$ are a set of $d \times d$ matrices whose components are:

$$\big[\Gamma^P_{\alpha\beta}\big]_{\mu\nu} \;=\; \delta_{\alpha\beta}\delta_{\mu\nu} - \frac{1}{2}\big(\delta_{\alpha\mu}\delta_{\beta\nu} + \delta_{\alpha\nu}\delta_{\beta\mu}\big) \; . \tag{17}$$

It should now be clear that, when considering the one loop effective action one needs to evaluate the expression:

$$\begin{aligned} \Gamma_P(A) \;&=\; -\int_0^\infty \frac{dT}{T} \, e^{-m^2 T} \int_{x(0)=x(T)} \mathcal{D}p\mathcal{D}x \; e^{i\int_0^T d\tau p(\tau)\cdot\dot{x}(\tau)} \\ &\times \;\; \mathrm{tr}\big[\Phi_P(T)\big] \;\; e^{-ie\int_0^T d\tau \dot{x}(\tau)\cdot A[x(\tau)]} \; . \end{aligned} \tag{18}$$

A quite remarkable property, that we may use to our advantage, is that the functional integral over x_μ, for a given background field A_μ is the same

for all the fields. The differences shall of course appear when evaluating the integrals over p_μ, since they are affected by the spin-dependent factor $\Phi(T)$.

We conclude this section by mentioning that the previous representations are not unique (in many ways). One of the reasons is that one may always describe a theory (with any spin) in terms of first order equations, although for a different set of field variables. Indeed, the equations of motion for a free field φ may always be written as follows [12]:

$$\big(\Gamma_\mu \partial_\mu + m\big)\phi(x) \;=\; 0 \tag{19}$$

where ϕ is a multicomponent field, defined in terms of φ and its derivatives, while Γ_μ are matrices whose form depend on the spin content of the original field φ. For example, for a massive real scalar field φ, we may write the action:

$$S \;=\; \frac{m}{2} \int d^dx\, \bar{\phi}(\Gamma_\mu \partial_\mu + m)\phi \;, \tag{20}$$

where the Γ_μ are, in this case, the $(d+1) \times (d+1)$ matrices

$$\big[\Gamma_\mu\big]_{ab} \;=\; \delta_{a,\mu+1}\delta_{\nu+1,b} \tag{21}$$

and

$$\phi \;=\; \begin{pmatrix} \varphi \\ -\partial_1\varphi/m \\ -\partial_2\varphi/m \\ \dots \\ -\partial_d\varphi/m \end{pmatrix} \;. \tag{22}$$

The 'adjoint' $\bar{\phi}$ is defined as: $\bar{\phi} = \phi^T \Gamma_0$, where

$$\Gamma_0 \;=\; \begin{pmatrix} 1 & 0_{1\times d} \\ 0_{d\times 1} & I_{d\times d} \end{pmatrix} \;. \tag{23}$$

If the field is instead complex, and it is coupled to an external gauge field A_μ, we have

$$S \;=\; m \int d^dx\, \bar{\phi}(\Gamma_\mu D_\mu + m)\phi \;, \tag{24}$$

with the only difference with respect to the previous case in that $\bar{\phi} = \phi^\dagger \Gamma_0$.

The generalization to higher spins J is simple, although care must be taken when considering $J > 1$ [12], due to the existence of non-trivial constraints on the state vectors, depending on representation chosen for the Γ_μ matrices.

Once this first-order formulation is introduced, one may write a world-line representation for the one-loop effective action Γ, which is given by:

$$\Gamma(A) \;=\; \int_0^\infty \frac{dT}{T}\, e^{-mT} \int_{x(0)=x(T)} \mathcal{D}p\,\mathcal{D}x\; e^{i\int_0^T d\tau p_\mu(\tau)\dot{x}_\mu(\tau)}$$

$$\times\; \mathrm{tr}\Big[\mathcal{P}e^{-i\int_0^T d\tau \Gamma_\mu p_\mu(\tau)}\Big]\; e^{-ie\int_0^T d\tau \dot{x}_\mu(\tau) A_\mu[x(\tau)]}\;, \tag{25}$$

which has the same structure as the one introduced for the Dirac case [1].

3 Constant external field in $1+1$ dimensions

3.1 Dirac field

We shall present here the evaluation of the fermionic determinant and propagator for a massive Dirac field in the presence of a constant external $F_{\mu\nu}$ field, in $1+1$ dimensions. As usual, rather than working directly with the determinant, we instead use the effective action $\Gamma_f(A)$,

$$\Gamma_f(A) \;=\; \int_0^\infty \frac{dT}{T}\, e^{-mT} \int_{x(0)=x(T)} \mathcal{D}p\,\mathcal{D}x\;\; e^{i\int_0^T d\tau p_\mu(\tau)\dot{x}_\mu(\tau)}$$

$$\times\; \mathrm{tr}\Big[\mathcal{P}e^{-i\int_0^T d\tau \not{p}(\tau)}\Big] e^{-ie\int_0^T d\tau\, \dot{x}_\mu(\tau) A_\mu(x(\tau))}, \tag{26}$$

where A_μ is such that:

$$F_{\mu\nu} \;=\; \partial_\mu A_\nu - \partial_\nu A_\mu \;=\; F\,\varepsilon_{\mu\nu}\;, \tag{27}$$

with $F \equiv$ constant. For A_μ we adopt a gauge-fixing condition such that

$$A_1(x) \;=\; -F\,x_2\;\;,\quad\; A_2 \;=\; 0\;. \tag{28}$$

We then see that:

$$\Gamma_f(A) \;=\; \int_0^\infty \frac{dT}{T}\, e^{-mT} \int \mathcal{D}p\;\; \mathrm{tr}\Big[\mathcal{P}e^{-i\int_0^T d\tau \not{p}(\tau)}\Big]\; \mathcal{Z}(p,F) \tag{29}$$

with

$$\mathcal{Z}(p,F) \;=\; \int_{x(0)=x(T)} \mathcal{D}x\;\; e^{i\int_0^T d\tau\, p_\mu(\tau)\dot{x}_\mu(\tau)}\;\; e^{ieF\int_0^T d\tau\, \dot{x}_1(\tau) x_2(\tau)}\;. \tag{30}$$

[1] The structure is more complicated for $J>1$, see [12].

We shall now evaluate $\mathcal{Z}(p,F)$. As it will become clear, the same object appears within the context of the complex scalar field determinant.

To evaluate, we first separate it into two iterated integrals, one for each component:

$$\mathcal{Z}(p,F) \;=\; \int_{x_2(0)=x_2(T)} \mathcal{D}x_2 \, \Big\{ e^{i\int_0^T d\tau \dot{x}_2(\tau)\, p_2(\tau)}$$

$$\times \int_{x_1(0)=x_1(T)} \mathcal{D}x_1 \; e^{i\int_0^T d\tau \dot{x}_1(\tau)[eFx_2(\tau)+p_1(\tau)]} \Big\} \;. \tag{31}$$

The two previous integrals are quite simple to evaluate, the result being:

$$\mathcal{Z}(p,F) \;\propto\; \exp[-\frac{i}{eF}\int_0^T d\tau\, \dot{p}_1(\tau) p_2(\tau)] \;, \tag{32}$$

an expression which captures the exact dependence on $p_\mu(\tau)$. However, in order to calculate $\Gamma_f(A)$ exactly, we need to know the exact form of $\mathcal{Z}(A)$, including any relevant global factor. One safe way to do that is, as usual, to introduce a discretization of the functional integral. For example, splitting the $[0,T]$ interval into n sub-intervals, we see that the functional integral over $x_1(\tau)$ is given by the limit:

$$\int_{x_1(0)=x_1(T)} \mathcal{D}x_1 \; e^{i\int_0^T d\tau \dot{x}_1(\tau)[eFx_2(\tau)+p_1(\tau)]}$$

$$= \lim_{n\to\infty} \left\{ \int \Big(\prod_{k=1}^n dx_1^{(k)} \Big) \; e^{i\sum_{k=1}^n (x_1^{(k+1)} - x_1^{(k)})[eFx_2^{(k)} + p_1^{(k)}]} \right\} \;, \tag{33}$$

where $x_1^{(k)}$ denotes $x_1(\tau)$ at the discrete time τ_k, with $\tau_k \equiv \frac{kT}{n}$, and a similar convention for x_2 and p_μ. Periodicity requires $x_\mu^{(n+1)} = x_\mu^{(1)}$. It is then immediate to see that:

$$\int \Big(\prod_{k=1}^n dx_1^{(k)} \Big) \; e^{i\sum_{k=1}^n (x_1^{(k+1)} - x_1^{(k)})[eFx_2^{(k)} + p_1^{(k)}]}$$

$$= \; L_1 \prod_{k=1}^{n-1} 2\pi\, \delta\Big(eF(x_2^{(l)} - x_2^{(l-1)}) \;+\; p_1^{(l)} - p_1^{(l-1)} \Big) \;, \tag{34}$$

where L_1 is the total length of the system along the x_1 coordinate. Discretizing also the $x_2(\tau)$ integral, an analogous calculation yields:

$$\begin{aligned}
\mathcal{Z}(p,F) &= L_1\, L_2 \lim_{n\to\infty} \left\{ (\frac{2\pi}{eF})^{n-1}\, e^{-\frac{i}{eF}\sum_{k=1}^{n} p_2^{(k)}(p_1^{(k+1)}-p_1^{(k)})} \right\} \\
&= \frac{eFL_1L_2}{2\pi} \lim_{n\to\infty} \left\{ (\frac{2\pi}{eF})^{n}\, e^{-\frac{i}{eF}\sum_{k=1}^{n} p_2^{(k)}(p_1^{(k+1)}-p_1^{(k)})} \right\} \\
&= \xi \lim_{n\to\infty} \left\{ (\frac{2\pi}{eF})^{n}\, e^{-\frac{i}{eF}\sum_{k=1}^{n} p_2^{(k)}(p_1^{(k+1)}-p_1^{(k)})} \right\} \;, \qquad (35)
\end{aligned}$$

where L_2 is the system length along the second coordinate. We have factored-out the dimensionless quantity $\xi \equiv \frac{eFL_1L_2}{2\pi}$, which measures the 'flux' through the system's area L_1L_2, in units of the elementary flux.

Note also that the product of δ functions implies, in particular, that the integral over $p_1(\tau)$ (to be performed next), will be over a space of periodic paths. Namely, the integral over x enforces periodic boundary conditions for the integral over p_1.

Then we insert the previous result for $\mathcal{Z}(p,T)$ into the expression for $\Gamma_f(A)$, and see that:

$$\begin{aligned}
\Gamma_f(A) \;&=\; \xi \int_0^\infty \frac{dT}{T}\, e^{-mT} \int_{p_1(0)=p_1(T)} \widehat{\mathcal{D}p} \\
&\times \operatorname{tr}\left[\mathcal{P} e^{-i\int_0^T d\tau \not{p}(\tau)}\right] e^{-\frac{i}{eF}\int_0^T d\tau\, \dot{p}_1(\tau) p_2(\tau)}, \qquad (36)
\end{aligned}$$

where the new integration measure for $p(\tau)$, $\widehat{\mathcal{D}p}$, is defined by:

$$\widehat{\mathcal{D}p} \;=\; \prod_{0<\tau\leq T} \frac{dp_1(\tau)dp_2(\tau)}{2\pi eF}, \qquad (37)$$

(note that one of the two (2π) factors from (7) cancels out). In the integral over $p(\tau)$, due to the presence of the term $\int d\tau \dot{p}_1(\tau)p_2(\tau)$ in the exponent, the functional integral is equivalent to the operatorial trace of an evolution operator, with the p_μ's replaced by time-independent, noncommuting operators:

$$\Gamma_f(A) \;=\; \xi \int_0^\infty \frac{dT}{T}\, e^{-mT}\, \operatorname{Tr}\Big(e^{-iT\hat{\not{p}}}\Big), \qquad (38)$$

where the $\hat{p}_\mu$'s satisfy the commutation relation:

$$[\hat{p}_1\,,\,\hat{p}_2] \;=\; -ieF \;, \qquad (39)$$

and the trace is over Hilbert and Dirac spaces.

To evaluate that trace we first write the operator $\not{\hat{p}}$ more explicitly, as follows:

$$\not{\hat{p}} \;=\; \sqrt{2eF}\,\widehat{\mathcal{O}}, \tag{40}$$

where

$$\widehat{\mathcal{O}} \;=\; \begin{pmatrix} 0 & \hat{a} \\ \hat{a}^\dagger & 0 \end{pmatrix}, \tag{41}$$

and $\hat{a} \equiv \frac{\hat{p}_1 - i\hat{p}_2}{\sqrt{2eF}}$, $\hat{a}^\dagger \equiv \frac{\hat{p}_1 + i\hat{p}_2}{\sqrt{2eF}}$ (we assume that $eF > 0$).

Since the operators $\hat{a}$ and $\hat{a}^\dagger$ verify $[\hat{a}, \hat{a}^\dagger] = 1$, we can calculate the spectrum of the self-adjoint operator $\widehat{\mathcal{O}}$ exactly. Indeed, we find the exact eigenvalues and eigenvectors to be the following:

$$\begin{aligned} \widehat{\mathcal{O}}|\varphi_n^{(\pm)}\rangle &= \lambda_n^{(\pm)}\,|\varphi_n^{(\pm)}\rangle\;,\quad n\in\mathbb{N} \\ \widehat{\mathcal{O}}|\varphi_0\rangle &= 0\;, \end{aligned} \tag{42}$$

where

$$\lambda_n^{(\pm)} \;=\; \pm\sqrt{n}\;,\quad n \;=\; 1,\,2,\,\ldots \tag{43}$$

and

$$\begin{aligned} |\varphi_n^{(\pm)}\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} \pm|n-1\rangle \\ |n\rangle \end{pmatrix} \quad n \;=\; 1,\,2,\,\ldots \\ |\varphi_0\rangle &= \begin{pmatrix} 0 \\ |0\rangle \end{pmatrix}. \end{aligned} \tag{44}$$

Here, $|n\rangle$ denotes the (normalized) eigenstates of the 'number' operator $\hat{a}^\dagger\hat{a}$. Note that the upper element in $|\varphi_0\rangle$ is 0 (the null vector), while the lower one is the 'vacuum' state.

Then the effective action becomes:

$$\Gamma_f(A) \;=\; \xi\int_0^\infty \frac{dT}{T}\,e^{-mT}\Big[1 \;+\; \sum_{n=1}^{\infty}\big(e^{-iT\sqrt{2eFn}} \;+\; e^{iT\sqrt{2eFn}}\big)\Big], \tag{45}$$

or, integrating out T:

$$\Gamma_f(A) \;=\; \xi\,\Big[\ln m + \sum_{n=1}^{\infty}\ln(m^2 + 2eFn)\Big]\;, \tag{46}$$

where we have neglected a constant which is independent of F and m.

Now to sum up the series, we use the representation:

$$\ln x \;=\; -\lim_{s\to 0}\frac{d}{ds}[x^{-s}], \tag{47}$$

to obtain:

$$\Gamma_f(A) \;=\; \xi \lim_{s\to 0} \frac{d}{ds} \left\{ \sum_{n=1}^{\infty} (m^2 + 2eFn)^{-s} + m^{-s} \right\} , \tag{48}$$

or:

$$\Gamma_f(A) \;=\; \xi \lim_{s\to 0} \frac{d}{ds} \left\{ \sum_{n=1}^{\infty} (2eF)^{-s} (n + \frac{m^2}{2eF})^{-s} + m^{-s} \right\} , \tag{49}$$

and

$$\Gamma_f(A) \;=\; \xi \lim_{s\to 0} \frac{d}{ds} \left\{ (2eF)^{-s} \zeta_H(s; 1 + \frac{m^2}{2eF}) + m^{-s} \right\} , \tag{50}$$

where ζ_H denotes Hurwitz ζ-function. The effective action is then obtained by taking the limit explicitly, and it is coincident with the results of [13], namely:

$$\begin{aligned} \Gamma_f(A) \;=\; L_1 L_2 \Big[& \frac{eF + m^2}{4\pi} \ln(2eF) \\ & + \frac{eF}{2\pi} \ln\Gamma(1 + \frac{m^2}{2eF}) - \frac{eF}{4\pi} \ln(2\pi m^2) \Big] . \end{aligned} \tag{51}$$

The imaginary part of $\tilde{\Gamma}_{(1+1)}(A)$ in Minkowski spacetime may be obtained by Wick rotating: $F \to iF$, so that:

$$\Im[\tilde{\Gamma}_{(1+1)}(A)] \;=\; \sum_{n=1}^{\infty} \arctan[\frac{2eFn}{m^2}] , \tag{52}$$

which can be also written in terms of the dimensionless vacuum angle [14] θ for the massive Schwinger model

$$\theta = \frac{2\pi F}{e}, \tag{53}$$

as:

$$\Im[\tilde{\Gamma}_{(1+1)}(A)] \;=\; \sum_{n=1}^{\infty} \arctan[(\frac{e^2}{m^2}) \, \frac{\theta n}{\pi}] . \tag{54}$$

The result for the imaginary part does not exhibit the periodicity in θ of the interacting model, since here the gauge field is not dynamical.

The procedure we have followed for the calculation of the effective action may of course also be applied to the propagator, if one takes into account the main differences: namely, that the integration over x is not over periodic paths, and that the spin degrees of freedom are not traced. Thus we are lead to:

$$G_f(x, y) \;=\; \int_0^{\infty} dT \, e^{-mT} \, \langle x | e^{-iT \not{p}} | y \rangle , \tag{55}$$

with the same definition for $\hat{p}$ we had in the effective action calculation. In abstract operatorial form:

$$G_f \;=\; \int_0^\infty dT\, e^{-mT}\; e^{-i\,T\,\hat{\not{p}}}\,, \tag{56}$$

and matrix elements may be taken with respect to any convenient basis. Since we already know the eigenvectors of $\hat{p}$, we can use that basis. Integrating out over the 'time' T the result is

$$G_f \;=\; \frac{1}{m}\mathcal{P}_0 \,+\, \sum_{n=1}^{\infty}\Big\{\frac{2m}{m^2+2eFn}\mathcal{P}_n \,+\, \frac{-2im}{m^2+2eFn}\mathcal{Q}_n\Big\} \tag{57}$$

where

$$\mathcal{P}_0 \;=\; \begin{pmatrix} 0 & 0 \\ 0 & |0\rangle\langle 0| \end{pmatrix} \tag{58}$$

$$\mathcal{P}_n \;=\; \begin{pmatrix} |n-1\rangle\langle n-1| & 0 \\ 0 & |n\rangle\langle n| \end{pmatrix} \quad (\forall n > 1)\;, \tag{59}$$

and

$$\mathcal{Q}_n \;=\; \begin{pmatrix} 0 & |n-1\rangle\langle n| \\ |n\rangle\langle n-1| & 0 \end{pmatrix} \quad (\forall n > 1)\;. \tag{60}$$

3.2 Complex scalar field

Let us now consider the changes that arise when calculating the effective action Γ_b, for the same gauge field configuration. First, we note that the calculation of $\mathcal{Z}(p,F)$ goes through in the same way as for the Dirac case, and we directly arrive to

$$\Gamma_b(A) \;=\; -\xi\,\int_0^\infty \frac{dT}{T}\, e^{-m^2T}\; \mathrm{Tr}\Big[e^{-T\,\hat{p}_\mu\hat{p}_\mu}\Big], \tag{61}$$

where the $\hat{p}_\mu$ operators are the same as the ones from the Dirac field calculation. The trace Tr is now over the Hilbert space only. In terms of the destruction and creation operators $\hat{a}$, $\hat{a}^\dagger$ used in the previous subsection, we see that:

$$\Gamma_b(A) \;=\; -\xi\,\int_0^\infty \frac{dT}{T}\, e^{-m^2T}\; \mathrm{Tr}\Big[e^{-T\,2eF(\hat{\mathcal{N}}+\frac{1}{2})}\Big], \tag{62}$$

where $\hat{\mathcal{N}}$ is the number operator corresponding to $\hat{a}$ and $\hat{a}^\dagger$:

$$\hat{\mathcal{N}} \;=\; \hat{a}^\dagger\hat{a}\;. \tag{63}$$

Then we write the trace in terms of the eigenvalues,

$$\Gamma_b(A) \;=\; -\xi \int_0^\infty \frac{dT}{T}\, e^{-m^2 T} \sum_{n=0}^{\infty} e^{-T\,2eF(n+\frac{1}{2})}, \tag{64}$$

and integrate over T to obtain:

$$\Gamma_b(A) \;=\; -\xi \sum_{n=0}^{\infty} \ln\left[m^2 \,+\, eF(2n+1)\right] . \tag{65}$$

Of course, this may be evaluated as in the Dirac case in terms of Hurwitz ζ function:

$$\Gamma_b(A) \;=\; \xi \frac{d}{ds}\Big[(2eF)^{-s}\,\zeta_H\big(s;\frac{1}{2}+\frac{m^2}{2eF}\big)\Big]\Big|_{s=0} . \tag{66}$$

Again, the result is identical to the one of [13].

The scalar propagator, G_b is simpler than its Dirac counterpart. Indeed, a straightforward calculation yields:

$$G_b \;=\; \int_0^\infty dT\, e^{-m^2 T}\; e^{-T\,2eF\,(\hat{a}^\dagger \hat{a}+\frac{1}{2})} , \tag{67}$$

or

$$G_b \;=\; \sum_{n=0}^{\infty} \frac{1}{m^2+2eF(n+\frac{1}{2})}\; |n\rangle\langle n| . \tag{68}$$

3.3 Complex Proca field

To calculate the effective action Γ_P (for the same gauge field configuration as before), we make again use of the result for $\mathcal{Z}(p,F)$, what in the present case leads to

$$\Gamma_P(A) \;=\; -\xi \int_0^\infty \frac{dT}{T}\, e^{-m^2 T}\; \mathrm{Tr}\Big[e^{-T\,\hat{p}_\alpha \hat{p}_\beta \Gamma^P_{\alpha\beta}}\Big], \tag{69}$$

with the same $\hat{p}_\mu$ operators as in the Dirac field case. The trace meant both over Hilbert space and Lorentz indices. In terms of the annihilation and creation operators $\hat{a}$, $\hat{a}^\dagger$ we have already introduced, we see that:

$$\Gamma_P(A) \;=\; -\xi \int_0^\infty \frac{dT}{T}\, e^{-m^2 T}\; \mathrm{Tr}\Big[e^{-\frac{eF}{2}\,T\,\hat{\mathcal{Q}}}\Big], \tag{70}$$

where $\hat{\mathcal{Q}}$ is the operator

$$\hat{\mathcal{Q}} \;=\; \begin{pmatrix} -(\hat{a}-\hat{a}^\dagger)^2 & i(\hat{a}^2-\hat{a}^{\dagger 2}) \\ i(\hat{a}^2-\hat{a}^{\dagger 2}) & (\hat{a}+\hat{a}^\dagger)^2 \end{pmatrix} . \tag{71}$$

In order to evaluate the trace, it is convenient to look for the eigenfunctions and eigenvalues of the $\mathcal{Q}$ operator. We first rewrite $\hat{\mathcal{Q}}$ as follows:

$$\hat{\mathcal{Q}} \;=\; (2\hat{\mathcal{N}}+1)\, I \,+\, \hat{a}^2\, \eta \,+\, (\hat{a}^\dagger)^2\, \eta^\dagger \;, \tag{72}$$

where I is the 2×2 identity matrix, while η denotes the nilpotent matrix:

$$\eta \;=\; \begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix} \;. \tag{73}$$

Eigenvalues λ and their corresponding eigenvectors $|\Psi\rangle$ of $\hat{\mathcal{O}}$ may be found, for example, by decomposing (an arbitrary) $|\Psi\rangle$ as follows:

$$|\Psi\rangle \;=\; |e_+\rangle \otimes |\chi_+\rangle \,+\, |e_-\rangle \otimes |\chi_-\rangle \tag{74}$$

where $|e_\pm\rangle$ are two-component vectors:

$$|e_\pm\rangle \;=\; \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix} \;, \tag{75}$$

which are obviously linearly independent and satisfy:

$$\begin{aligned} \eta|e_+\rangle \;=\; -2\,|e_-\rangle \quad &, \quad \eta^\dagger|e_-\rangle \;=\; -2\,|e_+\rangle \\ \eta^\dagger|e_+\rangle \;=\; 0 \quad &, \quad \eta|e_-\rangle \;=\; 0\;, \end{aligned} \tag{76}$$

while $|\chi_\pm\rangle$ are general Hilbert space vectors (scalars with respect to the Lorentz group).

Inserting the general decomposition into the eigensystem equation, we obtain:

$$\begin{aligned} \big(2\hat{\mathcal{N}}+1 \,-\, \lambda\big)\,|\chi_+\rangle \,+\, 2\,(\hat{a}^\dagger)^2|\chi_-\rangle \;&=\; 0 \\ 2\,\hat{a}^2|\chi_+\rangle \,+\, \big(2\hat{\mathcal{N}}+1 \,-\, \lambda\big)\,|\chi_-\rangle \;&=\; 0\,. \end{aligned} \tag{77}$$

Now it becomes trivial to solve the last system, for example by using the basis of eigenstates of the number operator for $|\chi_\pm\rangle$:

$$|\chi_\pm\rangle \;=\; \sum_{n=0}^{\infty} C_n^{(\pm)}\,|n\rangle \;, \tag{78}$$

where $|n\rangle$ denote the eigenvalues of the number operator. This yields recurrence relations for the $C_n^{(\pm)}$'s whose solutions are polynomials only if λ

equals an odd integer. Otherwise, the resulting eigenfunctions are not regular, and must therefore be discarded. For the regular solutions, $\lambda = 2l + 1$, $l = 0, 1, \ldots$, there is no degeneracy. Thus:

$$\Gamma_P(A) \;=\; -\xi \int_0^\infty \frac{dT}{T}\, e^{-m^2 T} \sum_{l=0}^{\infty} e^{-\frac{eF}{2}T\,(2l+1)} \;, \tag{79}$$

which may be integrated over T to obtain:

$$\Gamma_P(A) \;=\; -\xi \sum_{l=0}^{\infty} \ln\left[m^2 \,+\, \frac{eF}{2}(2l+1)\right] \,. \tag{80}$$

Of course, this is equivalent to the massive scalar field, with the trivial replacement: $eF \to \frac{eF}{2}$:

$$\Gamma_P(A) \;=\; \xi \frac{d}{ds}\Big[(eF)^{-s}\,\zeta_H\big(s; \frac{1}{2} + \frac{m^2}{eF}\big)\Big]\Big|_{s=0} \,. \tag{81}$$

4 Generalization to $d = 2k$ dimensions

The calculations of the previous section may be easily generalized to the case of a constant $F_{\mu\nu}$ field configuration in $d = 2k$ dimensions [2]. Indeed, one easily sees that the effective action for the fermionic case shall be given by an expression which is formally identical to (29)

$$\Gamma_f(A) \;=\; \int_0^\infty \frac{dT}{T}\, e^{-mT} \int \mathcal{D}p \;\mathrm{tr}\Big[\mathcal{P} e^{-i\int_0^T d\tau \not{p}(\tau)}\Big]\; \mathcal{Z}(p,F) \tag{82}$$

where $\mathcal{Z}(p,F)$ is given by:

$$\mathcal{Z}(p,F) \;=\; \int_{x(0)=x(T)} \mathcal{D}x \;\; e^{i\int_0^T d\tau \dot{x}_\mu(\tau) p_\mu(\tau)} \; e^{\frac{ie}{2}\int_0^T d\tau \dot{x}_\mu(\tau) F_{\mu\nu} x_\nu(\tau)} \;, \tag{83}$$

as follows from the gauge field configuration:

$$A_\mu(x) \;=\; -\frac{1}{2}\, F_{\mu\nu} x_\nu, \tag{84}$$

which satisfies the gauge-fixing condition $\partial \cdot A = 0$.

The easiest way to calculate $\mathcal{Z}(p,F)$ is to reduce the problem to a set of decoupled $1+1$-dimensional systems, and then to take advantage of the

[2] The essential features of the problem are the same for odd dimensions but they present subtleties which deserve a separate treatment [16].

results of the previous section. That may be done by using the fact that $\mathbf{F} \equiv (F_{\mu\nu})$ is a real antisymmetric matrix; hence it may be reduced to a block-diagonal form $\mathbf{f}$ by performing a similarity transformation with an orthogonal matrix $\mathbf{R}$:

$$\mathbf{F} \;=\; \mathbf{R}^T \, \mathbf{f} \, \mathbf{R} \,. \tag{85}$$

Each one of the blocks is 2×2 and antisymmetric, so that the reduced matrix has the following structure:

$$\mathbf{f} \;=\; \begin{pmatrix} 0 & f^{(1)} & 0 & 0 & 0 & \dots & 0 \\ -f^{(1)} & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & f^{(2)} & 0 & \dots & 0 \\ 0 & 0 & -f^{(2)} & 0 & 0 & \dots & 0 \\ \dots & & & & & & \\ 0 & 0 & 0 & \dots & 0 & 0 & f^{(k)} \\ 0 & 0 & 0 & \dots & 0 & -f^{(k)} & 0 \end{pmatrix} \,, \tag{86}$$

where the $f^{(a)}$, $(a \;=\; 1, \dots, k)$ are real numbers, which we assume to be different from zero (although the particular case of one or more of them being equal to zero may of course be dealt with at the end of the calculation). Then we redefine the momenta and coordinates in the path integral, according to the following transformation: $p_\mu \;\to\; (\mathbf{R}^{-1})_{\mu\nu} p_\nu$, $x_\mu \;\to\; (\mathbf{R}^{-1})_{\mu\nu} x_\nu$. The γ matrices are also redefined with $\mathbf{R}$ and of course we arrive to an equivalent representation of the Clifford algebra. We use the same notation for the new γ-matrices although we have the new representation in mind.

The general form of the matrix $\mathbf{F}$ can be further simplified in some particular cases, when there are some extra restrictions on the configuration. An interesting example corresponds to $d = 4$, where one has the possibility of considering a self-dual field:

$$\tilde{F}_{\mu\nu} \;=\; F_{\mu\nu} \;\;, \quad \tilde{F}_{\mu\nu} \;\equiv\; \frac{1}{2} \epsilon_{\mu\nu\rho\lambda} F_{\rho\lambda} \,. \tag{87}$$

This relation implies that $\mathbf{F}^2 \;=\; -f^2 \, \mathbf{I}$, where $\mathbf{I}$ is the unit matrix and $f^2 = \frac{1}{4} F_{\mu\nu} F_{\mu\nu}$. Then the two blocks in the canonical form for $\mathbf{F}$ are degenerate:

$$\mathbf{f} \;=\; \begin{pmatrix} 0 & f & 0 & 0 \\ -f & 0 & 0 & 0 \\ 0 & 0 & 0 & f \\ 0 & 0 & -f & 0 \end{pmatrix} \,, \tag{88}$$

what does simplify some calculations.

Rather than using the index μ, we introduce the notation $p_i^{(a)}$ ($a = 1, \ldots, k$, $i = 1, 2$), which distinguishes the components according to the 2×2 block they belong to. The same convention is adopted for x_μ. Then:

$$\Gamma_f(A) \;=\; \int_0^T \frac{dT}{T}\, e^{-mT} \int \mathcal{D}p \,\mathrm{tr}\Big[\mathcal{P} e^{-i\int_0^T d\tau \not{p}(\tau)}\Big]\; \mathcal{Z}(p, F)\;, \tag{89}$$

where

$$\mathcal{Z}(p, F) \;=\; \prod_{a=1}^{k} \int_{x^{(a)}(0)=x^{(a)}(T)} \mathcal{D}x^{(a)}\; \exp\Big\{ i \int_0^T d\tau \big[\dot{x}_i^{(a)}(\tau) p_i^{(a)}(\tau) + i\,\frac{e}{2} f^{(a)} \varepsilon_{ij} \dot{x}_i^{(a)}(\tau) x_j^{(a)}(\tau)\big] \Big\}\;. \tag{90}$$

Of course, for each value of a we have an integral which is identical to the one for the $1+1$ dimensional case. Thus:

$$\mathcal{Z}(p, F) \;=\; \prod_{a=1}^{k} \Big(\xi^{(a)} \lim_{n\to\infty} (\frac{2\pi}{ef^{(a)}})^n \Big\{ e^{-\frac{i}{ef^{(a)}} \sum_{k=1}^{n} p_2^{(k)} (p_1^{(k+1)} - p_1^{(k)})} \Big\} \Big)\;, \tag{91}$$

a result which we include into (89), to obtain:

$$\Gamma_f(A) \;=\; \Big[\prod_{a=1}^{k} \xi^{(a)}\Big] \int_0^\infty \frac{dT}{T}\, e^{-mT}$$

$$\int_{p_1^{(a)}(0)=p_1^{(a)}(T)} \prod_{a=1}^{k} \mathcal{D}\hat{p}^{(a)}\, \mathrm{tr}\Big\{ \mathcal{P} \exp\big[-i \int_0^T d\tau \gamma^{(a)} p_j^{(a)}(\tau)\big]\Big\}$$

$$\times \exp\big[-i \int_0^T d\tau \sum_{a=1}^{k} \frac{1}{ef^{(a)}} \dot{p}_1^{(a)}(\tau) p_2^{(a)}(\tau)\big]\;, \tag{92}$$

where:

$$\widehat{\mathcal{D}p^{(a)}} \;=\; \prod_{0<\tau\leq T} \frac{dp_1^{(a)}(\tau) dp_2^{(a)}(\tau)}{2\pi e f^{(a)}}\;. \tag{93}$$

Of course, the expression for $\Gamma_f(A)$ in (92) may be converted to:

$$\Gamma_f(A) \;=\; \Big[\prod_{a=1}^{k} \xi^{(a)}\Big] \int_0^\infty \frac{dT}{T}\, e^{-mT}\, \mathrm{Tr}\Big[e^{-iT\sum_{a=1}^{k} \hat{\not{p}}^{(a)}}\Big]\;, \tag{94}$$

where $\hat{\not{p}}^{(a)} \equiv \gamma_1^{(a)} \hat{p}_1^{(a)} + \gamma_2^{(a)} \hat{p}_2^{(a)}$ (a is not summed). The $\hat{p}^{(a)}$ operators verify the commutation relations:

$$[\hat{p}_j^{(a)}\,,\, \hat{p}_k^{(b)}] \;=\; -i\, f^{(a)}\, \delta^{ab}\, \varepsilon_{jk}\;, \tag{95}$$

and the trace is over Dirac and Hilbert space. Since the γ matrices satisfy the anticommutation relations:

$$\{\gamma_j^{(a)}\,,\,\gamma_k^{(b)}\} \;=\; 2\,\delta^{ab}\,\delta_{jk}\;, \tag{96}$$

we easily see that:

$$\not{p}^{(a)}\;\;\not{p}^{(b)} \;=\; 0\;\;,\;\;\forall a\neq b\;. \tag{97}$$

Then

$$\begin{aligned}\Gamma_f(A) \;&=\; \Big[\prod_{a=1}^{k}\xi^{(a)}\Big]\int_0^\infty\frac{dT}{T}\,e^{-mT}\\ \times\Big[1\,+&\sum_{n_1,\ldots,n_k=1}^{\infty}\sum_{a=1}^{k}\big(e^{-iT\sqrt{2ef^{(a)}n_a}}+e^{+iT\sqrt{2ef^{(a)}n_a}}\big)\Big]\;,\end{aligned} \tag{98}$$

or, doing the integral

$$\Gamma_f(A) \;=\; -\Big[\prod_{a=1}^{k}\xi^{(a)}\Big]\times\Big[\ln(m)+\sum_{n_1=1,\ldots,n_k=1}^{\infty}\ln\big(m^2+\sum_{a=1}^{k}(ef^{(a)}n_a)\big)\Big]\;. \tag{99}$$

which upon regularization leads to the known result ([13]).

5 Vortex-like background in $1+1$ dimensions

In this section we consider a singular background, corresponding to a gauge field A_μ such that:

$$\oint_{\mathcal{C}} dx_\mu\,A_\mu(x) \;=\; \int_{\mathcal{S}(C)} d^2x\,\varepsilon_{\mu\nu}\,\partial_\mu A_\nu \;=\; \Phi\,, \tag{100}$$

where $\mathcal{S}(C)$ is the planar region enclosed by $\mathcal{C}$, and Φ is a constant. In the symmetric gauge:

$$A_\mu(x) \;=\; -\frac{\Phi}{2\pi}\,\varepsilon_{\mu\nu}\,\frac{x_\nu}{x^2}\;. \tag{101}$$

Let us first consider a complex scalar field. In the expression that leads to its propagator, (9), we integrate out the p variable, obtaining:

$$\begin{aligned}G_b(x,y) \;&=\; \int_0^\infty dT\,e^{-m^2T}\int_{x(0)=y}^{x(T)=x}\mathcal{D}x\\ \times\;\exp-&\int_0^T d\tau\,\big[\frac{1}{4}\dot{x}^2(\tau)\,+\,i\,e\,\dot{x}_\mu(\tau)A_\mu(\tau)\big]\;,\end{aligned} \tag{102}$$

where we have absorbed a normalization constant in the integration measure for x. Next we rescale the time variable: $\tau \;=\; T\,t$, $0 \leq t < 1$ to derive the equivalent representation

$$G_b(x,y) \;=\; \int_0^\infty dT\, e^{-m^2 T} \int_{x(0)=y}^{x(1)=x} \mathcal{D}x$$

$$\times \,\exp - \int_0^1 dt\, \big[\frac{1}{4T}\dot{x}^2(t) \,+\, i\,e\,\dot{x}_\mu(t)A_\mu(t)\big] \;. \tag{103}$$

And we recognize (103) as the propagator for a non-relativistic particle of 'mass' $(2T)^{-1}$ moving in two spatial dimensions in a vortex-like background, corresponding to a time interval equal to $-i$. Recalling the results of ([15]), we may write an exact expression for the result of the path integral over $x_\mu(\tau)$:

$$\int_{x(0)=y}^{x(1)=x} e^{-\int_0^1 dt\,\left[\frac{1}{4T}\dot{x}^2(1)\,+\,i\,e\,\dot{x}_\mu(t)A_\mu(t)\right]} \;=\; \frac{e^{-\frac{1}{4T}(x^2+y^2)}}{4\pi T} \sum_{n=-\infty}^{+\infty} e^{-ien\Phi}$$

$$\times \int_{-\infty}^{+\infty} d\lambda\, e^{i\,\lambda\,\left(\theta_y-\theta_x+2\pi n\right)}\, I_\lambda\Big(\frac{|x||y|}{2T}\Big) \;, \tag{104}$$

where θ_x and θ_y are the angular coordinates of x and y, respectively, while I_λ is the modified Bessel function of order λ.

The sum over n can be transformed by means of Poisson's summation formula:

$$\sum_{n=-\infty}^{+\infty} e^{in(2\pi\lambda - e\Phi)} \;=\; \sum_{k=-\infty}^{+\infty} \delta\big(\lambda - \frac{e\Phi}{2\pi} + k\big) \;, \tag{105}$$

and this allows us to integrate out λ. Then:

$$G_b(x,y) \;=\; \int_0^\infty \frac{dT}{4\pi T}\, e^{-m^2 T}\, e^{-\frac{1}{4T}(x^2+y^2)} \,\times$$

$$\times \sum_{k=-\infty}^{+\infty} e^{i\left[k+\frac{e\Phi}{2\pi}(\theta_y-\theta_x)\right]}\; I_{k+\frac{e\Phi}{2\pi}}\Big(\frac{|x||y|}{2T}\Big) \;. \tag{106}$$

Following [15], we perform a unitary transformation in order to absorb the phase factor $e^{i\frac{e\Phi}{2\pi}(\theta_y-\theta_x)}$ into the propagator. At the level of fields, the transformation is

$$\phi(x) \;=\; e^{-i\frac{e\Phi}{2\pi}\,\theta_x}\;\tilde{\phi}(x)\;, \quad \bar{\phi}(y) \;=\; e^{i\frac{e\Phi}{2\pi}\,\theta_y}\;\tilde{\bar{\phi}}(y)\;, \tag{107}$$

while for the propagator we have:

$$\tilde{G}_b(x,y) \;\equiv\; \langle \tilde{\phi}(x)\, \bar{\tilde{\phi}}(y) \rangle \;=\; e^{i\frac{e\Phi}{2\pi}(\theta_x-\theta_y)}\, G_b(x,y)$$

$$= \int_0^\infty \frac{dT}{4\pi T}\, e^{-m^2 T}\, e^{-\frac{(x^2+y^2)}{4T}} \times \sum_{k=-\infty}^{+\infty} e^{ik(\theta_y-\theta_x)}\, I_{k+\frac{e\Phi}{2\pi}}\Big(\frac{|x||y|}{2T}\Big)\,, \tag{108}$$

and, of course, physical observables are not sensitive to this transformation.

Then we see that the form of the propagator $\tilde{G}_b(x,y)$ is periodic in the difference between the angles $\theta_y - \theta_x$, and it has the form of an expansion in 'partial waves':

$$\tilde{G}_b(x,y) \;=\; \sum_{k=-\infty}^{+\infty} e^{ik(\theta_y-\theta_x)}\, \tilde{G}_b^{(k)}(|x|,|y|) \tag{109}$$

where each $\tilde{G}_b^{(k)}$ corresponds to the k^{th} partial wave:

$$\tilde{G}_b^{(k)}(|x|,|y|) \;=\; \int_0^\infty \frac{dT}{4\pi T}\, e^{-m^2 T}\, e^{-\frac{(x^2+y^2)}{4T}} I_{k+\frac{e\Phi}{2\pi}}\Big(\frac{|x||y|}{2T}\Big)\,. \tag{110}$$

The integral over T cannot be performed exactly, but we may instead integrate term by term by expanding the Bessel function in a 'long-distance' approximation: $|x||y| >> T$. This amounts to large values of $|z|$ in the following asymptotic expansion

$$\begin{aligned} I_\nu(z) \;\;&\sim\;\; \frac{e^z}{(2\pi z)^{\frac{1}{2}}} \sum_{l=0}^{+\infty} \frac{(-1)^l}{(2z)^l}\, \frac{\Gamma(\nu+l+\frac{1}{2})}{l!\,\Gamma(\nu-l+\frac{1}{2})} \\ &+\;\; \frac{e^{-z+(\nu+1/2)i\pi}}{(2\pi z)^{\frac{1}{2}}} \sum_{l=0}^{+\infty} \frac{1}{(2z)^l}\, \frac{\Gamma(\nu+l+\frac{1}{2})}{l!\,\Gamma(\nu-l+\frac{1}{2})}\,. \end{aligned} \tag{111}$$

Inserting the first line in the previous expansion (the second is exponentially suppressed) for for each partial wave in (110), and integrating out T, we obtain:

$$\begin{aligned} \tilde{G}_b^{(k)}(|x|,|y|) \;&=\; \frac{1}{2\pi^{\frac{3}{2}}} \sum_{l=0}^{+\infty} \frac{(-1)^l}{l!}\, \frac{\Gamma(k+b+\frac{1}{2}+l)}{\Gamma(k+b+\frac{1}{2}-l)} \\ &\frac{1}{(|x||y|)^{l+1/2}}\, \Big(\frac{\big||x|-|y|\big|}{2m}\Big)^{l+\frac{1}{2}}\, K_{l+\frac{1}{2}}\big(m\big||x|-|y|\big|\big)\,, \end{aligned} \tag{112}$$

where we introduced the dimensionless flux: $b \equiv \frac{e\Phi}{2\pi}$.

6 Exact results for the massless Dirac field in $1+1$ dimensions

We will consider now the massless fermionic propagator in 1+1 dimension in the presence of an arbitrary external field. The result is known to be solvable [17]. We start with the fermionic propagator given by (4) with the vector field A_μ written in the form

$$A_\mu(x) = \partial_\mu \chi + i\epsilon_{\mu\nu}\partial_\nu \phi. \tag{113}$$

this is always possible in 1+1 dimension. The last factor in the path integral in Eq. (4) reads

$$-ie\int_0^T d\tau \dot{x}(\tau)\cdot A[x(\tau)] = -ie\int_0^T d\tau \dot{x}_\mu(\tau)\cdot(\partial_\mu\chi + i\epsilon_{\mu\nu}\partial_\nu\phi) \tag{114}$$

The first term gives a total derivative in the integral over the path which is independent of the path and can be taken out of the path integral

$$-ie\int_0^T d\tau \dot{x}(\tau)\cdot A[x(\tau)] = -ie(\chi(x)-\chi(y)) - ie\int_0^T d\tau \dot{x}_\mu(\tau) i\epsilon_{\mu\nu}\partial_\nu\phi \tag{115}$$

As was shown in [10, 11] terms with a $\dot{x}(\tau)$ can be substituted by a γ inside the path integral. Thus the path integral can be written in the following form

$$\begin{aligned} G_f(x,y,m) &= e^{-ie(\chi(x)-\chi(y))}\int_0^\infty dT\, e^{-mT}\int_{x(0)=y}^{x(T)=x} \mathcal{D}p\mathcal{D}x\; e^{i\int_0^T d\tau p(\tau)\cdot\dot{x}(\tau)} \\ &\times\; \mathcal{P}[e^{-i\int_0^T d\tau \not{p}(\tau) - e\gamma_5 \not{\partial}\phi}]\,, \end{aligned} \tag{116}$$

where we have used the fact that, in 2 dimensions,

$$i\gamma_\mu\epsilon_{\mu\nu}\partial_\nu = -\gamma_5\; \not{\partial}\,. \tag{117}$$

Consider now the action of the following infinitesimal transformation on a free fermionic propagator

$$G_f^\alpha(x,y,m) = e^{ie\gamma_5\alpha(x)}G_f(x,y,m)e^{ie\gamma_5\alpha(y)} \tag{118}$$

where $\alpha(x)$ is infinitesimal. Then the first order variation can be written as follows

$$\begin{aligned} \delta G_f^\alpha(x,y,m) &= G_f^\alpha(x,y,m) - G_f^0(x,y,m) \\ &= ie\gamma_5\alpha(x)G_f(x,y,m) + ieG_f(x,y,m)\gamma_5\alpha(y) \\ &= ie\gamma_5\alpha(x)G_f(x,y,m) - ie\gamma_5\alpha(y)G_f(x,y,-m). \end{aligned} \tag{119}$$

From the last equality it follows that for massless Dirac fields we have

$$\delta G_f^\alpha(x,y,m=0) = ie\gamma_5(\alpha(x)-\alpha(y))G_f(x,y,m=0). \tag{120}$$

The same will apply for the propagator in the presence of an external electromagnetic field, for the propagator in (122). Then we will have

$$\begin{aligned} G_f^\alpha(x,y,\phi) &= [1+ie\gamma_5(\alpha(x)-\alpha(y)]G_f(x,y,\phi) \\ &= e^{ie\gamma_5[\alpha(x)-\alpha(y)]}G_f(x,y,\phi) \end{aligned} \tag{121}$$

The factor can be introduced in the path integral, it appears as a term of the same form of the ϕ

$$\begin{aligned} G_f(x,y,m) &= e^{-ie(\chi(x)-\chi(y))}\int_0^\infty dT\, e^{-mT}\int_{x(0)=y}^{x(T)=x} \mathcal{D}p\mathcal{D}x\; e^{i\int_0^T d\tau p(\tau)\cdot\dot{x}(\tau)} \\ &\times\; \mathcal{P}\big[e^{-i\int_0^T d\tau \not{p}(\tau)-e\gamma_5\not{\partial}(\phi+\alpha)}\big]\,. \end{aligned} \tag{122}$$

One may then choose α to cancel the ϕ term. The final result is therefore:

$$G_f(x,y,\phi) = e^{-ie\gamma_5\phi(x)}G_f(x,y)e^{-ie\gamma_5\phi(y)}\;, \tag{123}$$

an expression could be useful in the numerical implementation of fermionic path integrals, which is a long-standing problem.

7 Anomalies and path roughness

The anomaly $\mathcal{A}$ corresponding to a global axial transformation of the Dirac fields:

$$\begin{aligned} \delta\psi &= i\,\xi\,\gamma_5\,\psi \\ \delta\bar{\psi} &= i\,\xi\,\bar{\psi}\gamma_5\,, \end{aligned} \tag{124}$$

is determined by the (functional and Dirac) regularized trace of γ_5,

$$\mathcal{A} \;=\; \lim_{M\to\infty} \mathrm{Tr}\Big[\gamma_5\, f(\frac{-\not{D}^2}{M^2})\Big] \tag{125}$$

where f is a regulating function which verifies the conditions:

$$f(0) \;=\; 1 \;\;,\quad f(\pm\infty) \;=\; f'(\pm\infty) \;=\; f''(\pm\infty) \;=\ldots = 0\,. \tag{126}$$

With the choice $f(x) = 1/(1+x^2)$, and after a little algebra we obtain the equivalent expression:

$$\mathcal{A} \;=\; \lim_{M\to\infty} M\,\mathrm{Tr}\Big[\gamma_5\,\big(\not{D}+M\big)^{-1}\Big]\,. \tag{127}$$

This of course may be written in terms of the propagator for a massive field, as follows:

$$\mathcal{A} \;=\; \lim_{M\to\infty} M \,\mathrm{tr}\left[\gamma_5\, G_f(x,x)\right] \tag{128}$$

where tr denotes the trace over spin indices only. Introducing the worldline representation, we see that

$$\begin{aligned}\mathcal{A} &= \lim_{M\to\infty} M \,\mathrm{tr}\left[\gamma_5\, G_f(x,x)\right] \\ &= \int_0^\infty dT\, e^{-mT} \int_{x(0)=x(T)} \mathcal{D}p\mathcal{D}x\; e^{i\int_0^T d\tau p(\tau)\cdot\dot{x}(\tau)} \\ &\times\;\; \mathrm{tr}\left[\gamma_5 \mathcal{P} e^{-i\int_0^T d\tau \not{p}(\tau)}\right]\, e^{-ie\int_0^T d\tau \dot{x}(\tau)\cdot A[x(\tau)]}\;. \end{aligned}\tag{129}$$

Then we note that γ_5 can, *inside the path integral*, be written purely in terms of $\dot{x}_\mu(\tau)$ and its derivatives, by using relations of the kind already introduced in [11]. There we related the average of γ_μ to $\dot{x}_\mu$. Now, with a precise form that shall depend on the number of spacetime dimensions, we can relate the average of γ_5 to (the average of) a product of $\dot{x}_\mu$ and its derivatives. Since the order in the product matters, we have to introduce a time splitting between the different factors.

To be more concrete, let us consider the case $d = 2$. Since $\gamma_5 = -i\gamma_1\gamma_2$,

$$\begin{aligned}\mathcal{A}_{d=2} &= -i \lim_{M\to\infty} M \int_0^\infty dT\, e^{-mT} \int_{x(0)=x(T)} \mathcal{D}p\mathcal{D}x\; e^{i\int_0^T d\tau p(\tau)\cdot\dot{x}(\tau)} \\ &\times\;\; \dot{x}_1(T)\dot{x}_2(T-\epsilon)\mathrm{tr}\left[\mathcal{P} e^{-i\int_0^T d\tau \not{p}(\tau)}\right]\, e^{-ie\int_0^T d\tau \dot{x}(\tau)\cdot A[x(\tau)]}\;, \end{aligned}\tag{130}$$

where ϵ is a positive infinitesimal parameter, to be taken to zero at the end of the calculation. Of course, we may use the more symmetric expression:

$$\begin{aligned}\mathcal{A}_{d=2} &= -\frac{i}{2} \lim_{M\to\infty} M \int_0^\infty dT\, e^{-mT} \int_{x(0)=x(T)} \mathcal{D}p\mathcal{D}x\; e^{i\int_0^T d\tau p(\tau)\cdot\dot{x}(\tau)} \\ &\times\;\; \varepsilon_{\mu\nu}\dot{x}_\mu(T)\dot{x}_\nu(T-\epsilon)\mathrm{tr}\left[\mathcal{P} e^{-i\int_0^T d\tau \not{p}(\tau)}\right]\, e^{-ie\int_0^T d\tau \dot{x}(\tau)\cdot A[x(\tau)]}\;, \end{aligned}\tag{131}$$

for which we use the notation:

$$\mathcal{A}_{d=2} \;=\; -\frac{i}{2} \lim_{M\to\infty} M \int_0^\infty dT\, e^{-mT} \left\langle \varepsilon_{\mu\nu}\dot{x}_\mu(T)\dot{x}_\nu(T-\epsilon)\right\rangle\;. \tag{132}$$

Expanding for small ϵ,

$$\mathcal{A}_{d=2} \;=\; \frac{i}{2} \lim_{M\to\infty} M \int_0^\infty dT\, e^{-mT}\, \epsilon \,\left\langle \varepsilon_{\mu\nu}\dot{x}_\mu(T)\ddot{x}_\nu(T)\right\rangle\;, \tag{133}$$

which shows that, in order to have a non-vanishing chiral anomaly in $d = 2$, the correlation function between $\dot{x}_\mu$ and $\ddot{x}_\nu$, has to be singular at the coincidence limit [3]. More precisely, the correlator between the velocity and the acceleration in orthogonal directions has to diverge as ϵ^{-1}. The object that appears inside the average symbol can also be given a geometric interpretation: indeed, if one uses the natural parameterization for the paths: $x_\mu = x_\mu(s)$ ($\Rightarrow\ \dot{x}^2 = 1$), then:

$$\frac{1}{2} \langle \varepsilon_{\mu\nu} \dot{x}_\mu(T) \ddot{x}_\nu(T) \rangle \ = \ \langle \frac{d\theta(s)}{ds} \rangle \tag{134}$$

where $\theta(s)$ is the angle between the tangent vector to the curve and a (fixed) reference line [4].

In this way, we can rephrase our statement about the roughness: $\langle \frac{d\theta(s)}{ds} \rangle$ is singular ($\sim \epsilon^{-1}$) when the anomaly is non-vanishing.

In $d = 4$, an entirely similar procedure yields

$$\mathcal{A}_{d=4} = -\frac{i}{6} \lim_{M\to\infty} M \int_0^\infty dT e^{-mT} \langle \varepsilon_{\mu\nu\rho\sigma} \dot{x}_\mu(T) \dot{x}_\nu(T-\epsilon) \dot{x}_\rho(T-2\epsilon) \dot{x}_\sigma(T-3\epsilon) \rangle \,, \tag{135}$$

which, in order for the lhs to be different from zero, requires a different kind of singularity in the correlation functions, since it involves up to to de fourth derivative of the curve. Its geometrical meaning is less appealing than in four dimensions (it is proportional to the rate of variation of the bi-torsion of the curve).

8 Conclusions

We have carried further the first-order spin formalism for the worldline, providing new tests and applications for Migdal's construction. We have thus obtained with this method, new expressions for the propagators and effective actions of the Dirac, Proca and complex scalar fields coupled to Abelian gauge fields. A remarkable result is the universality of the path order spin factor. In fact this can be seen as a natural consequence of the geometric representation and are extendible to higher spin fields.

For constant electromagnetic fields in two dimensions we have shown that our results agree with the results from the zeta-function renormalization. The results have been then generalized to d even dimensions. Notice that the first

[3] Similar qualitative results have been obtained in reference [18] for the effective action in the second order formalism.

[4] This object also appears in the 'Polyakov phase factor'. See, for example [19].

order formalism one can incorporate the quantum fluctuations of the gauge field. We have obtained an exact expression for the fermionic propagator in 1+1 dimensions. Other new results are the vortex configuration and the axial anomaly where the results can be understood in terms of a very attractive geometric feature of the contributing paths.

This can provide a useful contribution to the progress in nonperturbative quantum field dynamics within the worldline. Specially for its numerical implementation, which appears as a powerful new alternative [9]. Work in progress in odd dimensions indicates promising features of transferring internal degrees of freedom to geometrical properties of space time which could hopefully allow to include non Abelian fields.

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